OSCILLATIONS OF A BODY FILLED WITH A VISCOUS FLUID

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 7, No. 6, pp. 27–34, 1966

The oscillations of a physical pendulum containing a spherical cavity filled with an incompressible viscous liquid were discussed in [1]. In this paper we consider the more general problem of the motion of an axially symmetric solid with a spherical cavity filled with an incompressible viscous fluid and moving about a fixed point. It is assumed that the center of the cavity and the fixed point lie on the axis of symmetry of the body.

§1. BASIC EQUATIONS

1°. The relative motion of the liquid in the cavity inside the body is described by the Navier-Stokes equation,

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u},\nabla)\mathbf{u}+2(\boldsymbol{\Omega}\times\mathbf{u})+\frac{d\boldsymbol{\Omega}}{dt}\times\mathbf{r}+\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})=$$
$$=-\frac{1}{\gamma}\operatorname{grad}p-\nu\operatorname{rot}\operatorname{rot}\mathbf{u} \qquad (1.1)$$

and the continuity equation

$$\operatorname{div}\mathbf{u}=0. \qquad (1.2)$$

On the boundary of the cavity

$$\mathbf{u}=0, \qquad (1.3)$$

where $\mathbf{u}$ is the velocity of a fluid particle relative to the solid body, Ω is the angular velocity of the body, $\mathbf{r}$ is the position vector of the particle relative to the fixed point, γ is the density of the liquid, ν is the viscosity, and p is the pressure.

The equations of motion for the solid body can be obtained from the rate of change of the angular momentum of this system about the fixed point

$$d\mathbf{L}_0/dt=\mathbf{M}_0. \qquad (1.4)$$

The angular momentum of the system is given by

$$\mathbf{L}_0=\mathbf{L}_{01}+L_{02},$$

where $\mathbf{L}_{01}$ is the angular momentum of the body with the liquid solidified, and L_{02} is the angular momentum of the relative motion of the liquid; $\mathbf{M}_0$ is the principal moment of external forces acting on the system about the fixed point.

Let us suppose that the symmetry axis of the body executes small oscillations. We shall assume that the motion of the liquid inside the cavity is then also small.

2°. The motion of the liquid in the cavity will be referred to as a set of coordinates xyz rigidly attached to the solid. The center of the cavity O will be taken as the origin and the z-axis will be directed along the axis of symmetry of the body.

If we linearize Eq. (1.1) we obtain

$$\frac{\partial u_x}{\partial t}-2\Omega_z u_y+\frac{d\Omega_y}{dt}z-\frac{d\Omega_z}{dt}y=-\frac{\partial P}{\partial x}+\nu\Delta u_x,$$
$$\frac{\partial u_y}{\partial t}+2\Omega_z u_x-\frac{d\Omega_x}{dt}z+\frac{d\Omega_z}{dt}x=-\frac{\partial P}{\partial y}+\nu\Delta u_y,$$
$$\frac{\partial u_z}{\partial t}+\frac{d\Omega_x}{dt}y-\frac{d\Omega_y}{dt}x=-\frac{\partial P}{\partial z}+\nu\Delta u_z,$$

$$P=\frac{1}{\gamma}p+\left(\frac{d\Omega_y}{dt}x-\frac{d\Omega_x}{dt}y\right)b-$$
$$-\frac{1}{2}\Omega_z^2(x^2+y^2)+\Omega_z(\Omega_x x+\Omega_y y)(z+b), \qquad (1.5)$$

where b is the distance between the fixed point and the center of the cavity.

To determine the position of the body, consider two rectangular coordinate systems with the origin at the fixed point O, namely, a fixed system xyz with a vertical z-axis, and a semifixed system $\xi\eta\zeta$ which participates in the precessional and nutational motions of the body but does not take part in the rotation about the intrinsic axis. The ζ-axis of the semifixed system will be taken to lie along the symmetry axis of the body.

The position of the body can then be defined by the angle δ_1 between the axes η and y, the angle δ_2 between the ξ and x-axes, and the rotation angle δ_3 about the ζ axis.

Consider the motion of the system under the action of gravitational forces.

If we take the projections of Eq. (1.4) along the axes of the semifixed system, and linearize the resulting equations, we obtain

$$-A\delta_1''+C\Omega_z\delta_2'+\gamma\frac{d}{dt}\int_\tau[(xu_z-zu_x)\sin\delta_3+$$
$$+(yu_z-zu_y)\cos\delta_3]\,d\tau-$$
$$-\gamma b\frac{d}{dt}\int_\tau(u_y\cos\delta_3+u_x\sin\delta_3)\,d\tau=Qa\delta_1,$$
$$A\delta_2''+C\Omega_z\delta_1'+\gamma\frac{d}{dt}\int_\tau[-(xu_z-zu_x)\cos\delta_3+$$
$$+(yu_z-zu_y)\sin\delta_3]\,d\tau+$$
$$+\gamma b\frac{d}{dt}\int_\tau(u_x\cos\delta_3-u_y\sin\delta_3)\,d\tau=-Qa\delta_2,$$
$$\frac{d\Omega_z}{dt}+\frac{\gamma}{C}\frac{d}{dt}\int_\tau(xu_y-yu_x)\,d\tau=0,$$
$$(A=A_1+I+\gamma\tau b^2,\quad C=C_1+I,\quad \Omega_z=\Omega_\zeta), \qquad (1.6)$$

where A_1 and C_1 are the moments of inertia of the body with respect to the $\xi(\eta)$ and ζ axes, I is the moment of inertia of the liquid about the diameter of the cavity, τ is the volume of the cavity, Q is the total weight of the system, and a is the distance between the fixed point and the center of mass of the system which lies below the fixed point.

Since the center of mass of the liquid lies at the origin of the fixed system of coordinates, the component of the relative momentum of the liquid along this axis must be zero, i.e.,

$$\gamma\int_\tau u_x\,d\tau=0,\qquad \gamma\int_\tau u_y\,d\tau=0. \qquad (1.7)$$

§2. SOLUTION IN TERMS OF GENERALIZED SPHERICAL COORDINATES

1°. The solution of the problem will be sought in the form of a series in terms of the generalized spherical functions [2]. With this aim in view let us transform in Eqs. (1.2), (1.5), and (1.6) to the spherical coordinates ρ, θ, φ, and introduce the complex velocity combinations

$$u_+ = -{}^1/_2\sqrt{2}\,(u_\varphi + iu_\theta),$$
$$u_0 = u_\rho, \quad u_- = {}^1/_2\sqrt{2}\,(u_\varphi - iu_\theta).$$

Using Eqs. (1.7) we then obtain the following set of equations describing the motion of the liquid:

$$\frac{\partial u_0}{\partial\rho} + \frac{2u_0}{\rho} + \frac{i}{\rho\sqrt{2}}\left(\frac{\partial u_+}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial u_+}{\partial\varphi} + u_+\operatorname{ctg}\theta\right) + \frac{i}{\rho\sqrt{2}}\left(\frac{\partial u_-}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial u_-}{\partial\varphi} + u_-\operatorname{ctg}\theta\right) = 0\,, \qquad (2.1)$$

$$\frac{\partial u_0}{\partial t} - \sqrt{2}\Omega_z u_-\sin\theta + \sqrt{2}\Omega_z u_+\sin\theta = -\frac{\partial P}{\partial\rho} + \nu\Big[\frac{\partial^2 u_0}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial u_0}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u_0}{\partial\theta^2} + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2 u_0}{\partial\varphi^2} + \frac{\operatorname{ctg}\theta}{\rho^2}\frac{\partial u_0}{\partial\theta} - \frac{2u_0}{\rho^2} - \frac{i\sqrt{2}}{\rho^2}\left(\frac{\partial u_+}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial u_+}{\partial\varphi} + u_+\operatorname{ctg}\theta\right) - \frac{i\sqrt{2}}{\rho^2}\left(\frac{\partial u_-}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial u_-}{\partial\varphi} + u_-\operatorname{ctg}\theta\right)\Big],$$

$$\frac{\partial u_+}{\partial t} - \sqrt{2}\Omega_z u_0\sin\theta - 2i\Omega_z u_+\cos\theta + \frac{\rho}{\sqrt{2}}\frac{\partial\Omega_x}{dt}(\cos\theta\cos\varphi + i\sin\varphi) + \frac{\rho}{\sqrt{2}}\frac{d\Omega_y}{dt}(\cos\theta\sin\varphi - i\cos\varphi) - \frac{\rho}{\sqrt{2}}\sin\theta\frac{d\Omega_z}{dt} = \frac{i}{\rho\sqrt{2}}\left(\frac{\partial P}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial P}{\partial\varphi}\right) + \nu\Big[\frac{\partial^2 u_+}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial u_+}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u_+}{\partial\varphi^2} + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2 u_+}{\partial\varphi^2} + \frac{\operatorname{ctg}\theta}{\rho^2}\frac{\partial u_+}{\partial\theta} - \frac{u_+}{\rho^2\sin^2\theta} - \frac{2i\cos\theta}{\rho^2\sin^2\theta}\frac{\partial u_+}{\partial\varphi} - \frac{i\sqrt{2}}{\rho^2}\left(\frac{\partial u_0}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial u_0}{\partial\varphi}\right)\Big],$$

$$\frac{\partial u_-}{\partial t} + \sqrt{2}\Omega_z u_0\sin\theta + 2i\Omega_z u_-\cos\theta + \frac{\rho}{\sqrt{2}}\frac{d\Omega_x}{dt}(-\cos\theta\cos\varphi + i\sin\varphi) - \frac{\rho}{\sqrt{2}}\frac{d\Omega_y}{dt}(\cos\theta\sin\varphi + i\cos\varphi) + \frac{\rho}{\sqrt{2}}\sin\theta\frac{d\Omega_z}{dt} = \frac{i}{\rho\sqrt{2}}\left(\frac{\partial P}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial P}{\partial\varphi}\right) + \nu\Big[\frac{\partial^2 u_-}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{2}{\rho}\frac{\partial u_-}{\partial\rho} + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2 u_-}{\partial\varphi^2} + \frac{\operatorname{ctg}\theta}{\rho^2}\frac{\partial u_-}{\partial\theta} - \frac{u_-}{\rho^2\sin^2\theta} + \frac{2i\cos\theta}{\rho^2\sin^2\theta}\frac{\partial u_-}{\partial\varphi} - \frac{i\sqrt{2}}{\rho^2}\left(\frac{\partial u_0}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial u_0}{\partial\varphi}\right)\Big],$$

$$\alpha = \Omega_y - i\Omega_x\,. \qquad (2.2)$$

If we now introduce the complex quantity $\beta = \delta_2 + i\delta_1$, the equations of motion for the body will assume the form

$$A\beta'' - iC\Omega_z\beta' + Q\alpha\beta + i\gamma\frac{d}{dt}\left\{e^{i\delta_3}\int_\tau e^{i\varphi}\left[\frac{u_+}{\sqrt{2}}(1-\cos\theta) + \frac{u_-}{\sqrt{2}}(1+\cos\theta)\right]\right\}\rho\,d\tau = 0\,, \qquad (2.3)$$

$$\frac{d\Omega_z}{dt} + \frac{\gamma}{C}\frac{d}{dt}\int_\tau \frac{u_- - u_+}{\sqrt{2}}\,p\sin\theta\,d\tau = 0. \qquad (2.4)$$

2°. The solution of the hydrodynamic problem will be sought in the form

$$u_0 = \sum_{l=1}^{\infty}\sum_{n=-l}^{l} f_{0n}^{\,l}(\rho, t)\,T_{0n}^{\,l}({}^1/_2\pi - \varphi, \theta, 0),$$
$$u_\pm = \sum_{l=1}^{\infty}\sum_{n=-l}^{l} f_{\pm1n}^{\ \ l}(\rho, t)\,T_{\pm1n}^{\,l}({}^1/_2\pi - \varphi, \theta, 0),$$
$$P = \sum_{l=0}^{\infty}\sum_{n=-l}^{l} F_n^{\,l}(\rho, t)\,T_{0n}^{\,l}({}^1/_2\pi - \varphi, \theta, 0), \qquad (2.5)$$

where f_{0n}^l, $f_{\pm1n}^l$, F_n^l are unknown functions of ρ and t, and T_{0n}^l, $T_{\pm1n}^l$ are the generalized spherical functions [2].

Let us substitute the series (2.5) into Eqs. (2.1)–(2.6) and equate the coefficients of the same spherical functions. After some rearrangement we have

$$\frac{\partial f_{0n}^{\ l}}{\partial\rho} + \frac{2}{\rho}f_{0n}^{\ l}\,\frac{1}{\rho} + \left(\frac{l(l+1)}{2}\right)^{1/2}(f_{1n}^{\ l} + f_{-1n}^{\ l}) = 0, \quad (2.6)$$

$$\frac{\partial f_{0n}^{\ l}}{\partial t} + i\sqrt{2}\Omega_z iA(l)(f_{1n}^{\ l} - f_{-1n}^{\ l}) + \sqrt{2}\frac{\Omega_z in}{\sqrt{l(l+1)}}(f_{1n}^{\ l} + f_{-1n}^{\ l}) - \sqrt{2}\Omega_z iB(l)\left(\frac{l+2}{l}\right)^{1/2}(f_{1n}^{\ l+1} - f_{-1n}^{\ l+1}) = \frac{\partial F_n^{\ l}}{\partial\rho} + \nu\left[\frac{\partial^2 f_{0n}^{\ l}}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial f_{0n}^{\ l}}{\partial\rho} - \frac{2 + l(l+1)}{\rho^2}f_{0n}^{\ l} - \frac{\sqrt{2l(l+1)}}{\rho^2}(f_{1n}^{\ l} + f_{-1n}^{\ l})\right], \qquad (2.7)$$

$$\frac{\partial}{\partial t}(f_{1n}^{\ l} + f_{-1n}^{\ l}) + 2\sqrt{2}\frac{\Omega_z in}{\sqrt{l(l+1)}}f_{0n}^{\ l} - 2\Omega_z\frac{in}{l(l+1)}(f_{1n}^{\ l} + f_{-1n}^{\ l}) - 2\Omega_z iA(l)\left(\frac{l+1}{l}\right)^{1/2}(f_{1n}^{\ l-1} - f_{-1n}^{\ l-1}) - 2\Omega_z iB(l)\left(\frac{l+2}{l+1}\right)^{1/2}(f_{1n}^{\ l+1} - f_{-1n}^{\ l+1}) = \frac{\sqrt{2l(l+1)}}{\rho}F_n^{\ l} + \nu\left[\frac{\partial^2}{\partial\rho^2}(f_{1n}^{\ l} + f_{-1n}^{\ l}) + \frac{2}{\rho}\frac{\partial}{\partial\rho}(f_{1n}^{\ l} + f_{-1n}^{\ l}) - \frac{l(l+1)}{\rho^2}(f_{1n}^{\ l} + f_{-1n}^{\ l}) - \frac{2}{\rho}\sqrt{2l(l+1)}\,f_{0n}^{\ l}\right], \quad (2.8)$$

$$\frac{\partial}{\partial t}(f_{1n}^{\ l} - f_{-1n}^{\ l}) - 2\Omega_z i\frac{n}{l(l+1)}(f_{1n}^{\ l} - f_{-1n}^{\ l}) - 2\sqrt{2}\Omega_z iA(l)f_{0n}^{\ l-1} - 2\Omega_z iA(l)\left(\frac{l+1}{l}\right)^{1/2}(f_{1n}^{\ l-1} - f_{-1n}^{\ l-1}) + 2\sqrt{2}\Omega_z iB(l)f_{0n}^{\ l+n} - 2\Omega_z iB(l)\left(\frac{l+2}{l+1}\right)^{1/2}(f_{1n}^{\ l+1} + f_{-1n}^{\ l+1}) - \delta_0^{\ l-1}i2\rho\frac{d\Omega_z}{dt} + \delta_{-1n}^{\ \ -l}\frac{\rho}{\sqrt{2}}\frac{d\alpha}{dt} + \delta_{1n}^{\ l}\frac{\rho}{\sqrt{2}}\frac{d\bar{\alpha}}{dt} =$$

$$= \nu \left[\frac{\partial^2}{\partial \rho^2} (f_{1n}{}^l - f_{-1n}{}^l) + \frac{2}{\rho} \frac{\partial}{\partial \rho} (f_{1n}{}^l - f_{-1n}{}^l) - \right.$$

$$\left. - \frac{2}{\rho^2} (f_{1n}{}^l - f_{-1n}{}^l) \right], \quad A(l) = \left(\frac{(l^2 - n^2)(l-1)}{(2l-1)^2 l} \right)^{1/2},$$

$$B(l) = \left(\frac{[(l+1)^2 - n^2] \, l}{(2l+3)^2 (l+1)} \right)^{1/2}, \qquad (2.9)$$

$$A\beta'' - iC\Omega_z\beta' + Q\alpha\beta -$$

$$- \gamma \frac{8\sqrt{2}}{3} \pi \frac{d}{dt} \int_0^R e^{i\delta_3} (f_{1-1}{}^1 - f_{-1-1}{}^1) \rho^3 d\rho = 0, \quad (2.10)$$

$$\frac{d\Omega_z}{dt} + i \frac{4}{3} \frac{\gamma}{C} \frac{d}{dt} \int_0^R (f_{10}{}^1 - f_{-10}{}^1) \rho^3 d\rho = 0, \quad (2.11)$$

$$(l = 1, 2, 3, \ldots, \infty;$$

$$n = -l, -l+1, \ldots, 0, 1, 2, \ldots, l).$$

Equations (2.8)–(2.9) were obtained by adding and subtracting the equations corresponding to the second and third equations in Eqs. (2.2). In these expressions R is the radius of the cavity, δ^l_{mn} is equal to 1 if all three indices are equal and 0 in all other cases. The condition given by Eq. (1.3) will be satisfied if

$$f_{0n}^l(R, t) = f_{\mp 1n}^l(R, t) = 0. \qquad (2.12)$$

Equations (2.10)–(2.11) show that the motion of the body is affected only by those motions of the fluid which are described by the terms of the series (2.5) with $l = 1$ and $n = -1, 0$.

§3. THE CASE OF A SLOW PROPER ROTATION

Consider the motion of the body with a small angular velocity Ω_z about the z-axis.

We shall assume that the terms in Eqs. (2.6)–(2.9) which contain the product $\Omega_z f_{mn}{}^l$ can be neglected. All equations with $l = 1$ except for Eqs. (2.9) will then be the same as the corresponding equations for the oscillations of a liquid in a fixed vessel [3].

The solutions of Eqs. (2.9) with $l = 1$ can be sought as in [1] in the form of series in terms of the eigenfunctions for this problem, i.e.,

$$f_{1n}{}^1 - f_{-1n}{}^1 = 2 \sum_{j=1}^{\infty} \Phi_j(s_j) \, C_{1n}{}^j(t), \quad \Phi_j(s_j) = \frac{J_{3/2}(s_j\rho / R)}{\sqrt{s_j\rho / R}}. \quad (3.1)$$

These functions satisfy the boundary condition (2.12); C^f_{1n} are unknown functions of time and s_j are the roots of the Bessel function $J_{3/2}(s)$.

To determine Ω_z, let us solve Eqs. (2.11) and (2.9) with $l = 1$ and $n = 0$:

$$\frac{\partial}{\partial t} (f_{10}{}^1 - f_{-10}{}^1) - i2\rho \frac{d\Omega_z}{dt} =$$

$$= \nu \left[\frac{\partial^2}{\partial \rho^2} (f_{10}{}^1 - f_{-10}{}^1) + \frac{2}{\rho} \frac{\partial}{\partial \rho} (f_{10}{}^1 - f_{-10}{}^1) - \frac{2}{\rho^2} (f_{10}{}^1 - f_{-10}{}^1) \right]. \quad (3.2)$$

Substituting Eqs. (3.1) into Eqs. (2.11) and (3.2) and replacing ρ in Eq. (3.2) by the expansion

$$\rho = -\sqrt{2\pi} R \sum_{j=1}^{\infty} \frac{\sqrt{1 + s_j^2}}{s_j} \Phi_j(s_j),$$

we have, if we equate the coefficients of the same Φ_j,

$$\frac{dC_{10}{}^j}{dt} + \nu \left(\frac{s_j}{R} \right)^2 C_{10}{}^j + iR\sqrt{2\pi} \frac{\sqrt{1 + s_j^2}}{s_j} \frac{d\Omega_z}{dt} = 0,$$

$$\frac{d\Omega_z}{dt} - i \frac{\gamma}{C} \frac{8\sqrt{2\pi}}{3} \sum_{j=1}^{\infty} \frac{1}{s_j \sqrt{1 + s_j^2}} \frac{dC_{10}{}^j}{dt} = 0. \qquad (3.3)$$

This is a set of ordinary differential equations. We shall seek its solution proportional to $e^{\varkappa t}$. In this way we obtain the characteristic equation:

$$0.1 \frac{C - I}{I} \varkappa = -\varkappa \sum_{j=1}^{\infty} \frac{1}{s_j^2 + R^2 \nu^{-1} \varkappa}. \qquad (3.4)$$

It is readily shown by a graphical method that this equation has one zero root and a denumerable set of real negative roots. Therefore, if Ω_z is small at the initial instant of time, it will remain small for the remainder of the time.

To find β let us substitute Eq. (3.1) into Eqs. (2.10) and (2.9), and assume in the last equation that $l = 1$, $n = -1$. Calculations similar to those used to determine Ω_z lead to the following set of ordinary differential equations:

$$A\beta'' + Q\alpha\beta + \gamma \frac{16\sqrt{\pi}}{3} R^4 \sum_{j=1}^{\infty} \frac{1}{s_j \sqrt{1 + s_j^2}} \frac{d}{dt} [e^{i\delta_3} C_{1-1}{}^j] = 0, \quad (3.5)$$

$$\frac{dC_{1-1}{}^j}{dt} + \frac{\nu}{R^2} s_j^2 C_{1-1}{}^j = \sqrt{\pi} R \frac{\sqrt{1 + s_j^2}}{s_j} \frac{d\alpha}{dt}. \qquad (3.6)$$

Let us consider the solutions of these two equations which are proportional to $e^{\lambda t}$ in the case when Ω_z = const (so that $\delta_3 \approx \Omega_z t$).

The characteristic equation is now

$$(A - I)\lambda^2 + Q\alpha + 10I\lambda^2 \sum_{j=1}^{\infty} \frac{1}{s_j^2 + R^2\nu^{-1}(\lambda - i\Omega_z)} = 0. \quad (3.7)$$

§4. THE CASE OF AN ARBITRARY ROTATION

This problem becomes much more complicated if we remove the restriction on the magnitude of Ω_z. In fact, the function $f_{1n}^1 - f_{-1n}^1$ in Eqs. (2.10) and (2.11) will be one of the unknown functions in an infinite set of coupled Eqs. (2.9) with odd l, and Eqs. (2.6)–(2.8) with even l. Any finite number of equations of this system contains more unknowns than the number of equations. We have, however, succeeded in finding a special solution of these systems.

Let us try to find the solution of Eqs. (2.6)–(2.11) by substituting $f_{0n}^2 = 0$. From Eq. (2.6) we find that for $l = 2$

$$f_{1n}{}^2 + f_{-1n}{}^2 = 0.$$

From the structure of Eqs. (2.6)–(2.9) we can see that the functions f^l_{mn} with $l > 1$ are determined. However, there is no need to determine these functions to find Ω_z and β.

To find Ω_z we must solve, as in the first problem, the equations given by (2.11) and (2.9) with $l = 1$ and $n = 0$. If we substitute $f_{0n}^2 = f_{-1n}^2 + f_{1n}^2 = 0$ in Eq. (2.9) we obtain Eq. (3.2), and hence Ω_z is found in the same way as before.

Let us now find β in the case in which Ω_z is a constant. Here we have $\delta_3 \approx \Omega_z t$ (we are assuming that the initial angle of rotation of the body about the ζ axis is zero).

To find β let us substitute Eq. (3.1) into Eqs. (2.10) and (2.9), and assume in the last equation that

$$l = 1, \; n = -1, \; f_{0n}{}^2 = f_{1n}{}^2 + f_{-1n}{}^2 = 0 \; \text{ and } \; \delta_3 = \Omega_z t.$$

Separation of variables as in the previous problems leads to the following set of ordinary differential equations:

$$A\beta'' - iC\Omega_z\beta' + Q\alpha\beta -$$

$$- \gamma \frac{16\sqrt{\pi}}{3} R^4 \sum_{j=1}^{\infty} \frac{1}{s_j^2 \sqrt{1 + s_j^2}} \frac{d}{dt} (e^{i\delta_3} C_{1-1}{}^j) = 0, \quad (4.1)$$

$$\frac{dC_{1-1}{}^j}{dt} + \left(\frac{\nu}{R^2} s_j^2 + i\Omega_z \right) C_{1-1}{}^j = \sqrt{\pi} R \frac{\sqrt{1 + s_j^2}}{s_j} \frac{d\alpha}{dt}, \quad (4.2)$$

$$(A - J)\lambda^2 - i(C - J)\Omega_z\lambda + Q\alpha +$$

$$+ \lambda(\lambda - i\Omega_z) 10J \sum_{j=1}^{\infty} \frac{1}{s_j^2 + R^2\nu^{-1}\lambda} = 0. \qquad (4.3)$$

§ 5. THE APPROXIMATE FREQUENCY EQUATIONS FOR LARGE AND SMALL VISCOSITIES

Let us now introduce the dimensionless parameter ε and derive the approximate equations for the frequencies when ε and ε^{-1} are small. Equation (3.7) can be reduced to the form

$$\left(1-\frac{J}{A}\right)\mu^2+1=-10\frac{J}{A}\mu^2 S^* \qquad \left(S^*=\sum_{j=1}^{\infty}\frac{1}{s_j^2+\varepsilon(\mu-i\omega)}\right)$$

$$\lambda=\mu\left(\frac{Qa}{A}\right)^{1/2}, \qquad \Omega_z=\omega\left(\frac{Qa}{A}\right)^{1/2}, \qquad \varepsilon=\frac{R^2}{\nu}\left(\frac{Qa}{A}\right)^{1/2}. \quad (5.1)$$

Let us now assume that ε is small. If we subtract the infinite series $(s_1^{-2}+s_2^{-2}+s_3^{-2}+\ldots)$ from the right side of Eq. (3.7), and then add the same number, and if we also add and subtract the following quantities

$$-\varepsilon(\mu-i\omega)\sum_{j=1}^{\infty}s_j^{-4}, \quad \varepsilon^2(\mu-i\omega)^2\sum_{j=1}^{\infty}s_j^{-6}, \quad -\varepsilon^3(\mu-i\omega)^3\sum_{j=1}^{\infty}s_j^{-8},$$

and so on, then Eq. (5.1) can be transformed to read

$$\mu^2+1=10\frac{J}{A}\mu^2[\varepsilon(\mu-i\omega)S_4-$$
$$-\varepsilon^2(\mu-i\omega)^2S_6+\varepsilon^3(\mu-i\omega)^3S_8-\ldots]$$
$$\left(S_{2N}=\sum_{j=1}^{\infty}s_j^{-2N}\right). \quad (5.2)$$

The numbers s_j^2 in the interval [0, 1] will be the eigenvalues of the integral operator with the kernel

$$K(x,y)=\begin{cases}{}^1\!/_2\,x^3y-1.8xy+{}^1\!/_2\,xy^3+x & (x\leqslant y)\\ {}^1\!/_2\,x^3y-1.8xy+{}^1\!/_2\,xy^3+y & (x\geqslant y).\end{cases}$$

Using the well-known theorems of the theory of integral equations, we can show that

$$S_2=0.1, \quad S_4=\frac{1}{350}, \quad S_6=\frac{1}{3^25^37}, \quad S_8=0.6089\cdot10^{-5}\text{ etc.}$$

Equation (5.2) then assumes the form

$$\mu^2+1=JA^{-1}[\varepsilon(\mu-i\omega)\,0.02857-$$
$$-\varepsilon^2(\mu-i\omega)^2,\ 0.127\cdot10^{-2}+$$
$$+\varepsilon^3(\mu-i\omega)\,0.6089\cdot10^{-4}+\ldots]. \quad (5.3)$$

We shall seek μ in the form

$$\mu=\mu_0+\varepsilon\mu_1+\varepsilon^2\mu_2+\ldots. \quad (5.4)$$

Substituting this series into Eq. (5.3), and equating the coefficients of equal powers of ε, we find the following relations for μ:

$$\mu=\pm i-\varepsilon\,0.01429\frac{J}{A}(1\mp\omega)+$$
$$+\varepsilon^2 i\left[0.5.102\left(\frac{J}{A}\right)^2(\mp1+1.6\,\omega\mp0.6\,\omega^2)+\right.$$
$$\left.+0.6349(\pm1-2\omega\pm\omega^2)\right]10^{-3}+$$
$$+\varepsilon^3\left[2.332\left(\frac{J}{A}\right)^3(1+2\,\omega^2\mp1.312\,\omega\mp0.6875\,\omega^3)-\right.$$
$$-5.442\left(\frac{J}{A}\right)^2(1\mp0.1667\,\omega-2.333\,\omega^2\mp0.1667\,\omega^3)+$$
$$\left.+3.044\frac{J}{A}(1\mp\omega)^3\right\}10^{-5}+\ldots$$

Now suppose that ε is small. If we replace the root of the Bessel function $J_{3/2}(s)$ by the first terms of the series expansion, we can find the approximate value of the sum S^* in (5.1).

We have $s_j=a_j-\alpha_j$, where $a_j=(\pi/2)(1+2_j)$,

$$\alpha_j=\frac{1}{a_j}+\frac{2}{3a_j^3}+\frac{13}{15a_j^5}+\cdots \quad (j=1,2,3,\ldots), \quad (5.5)$$

$$\sum_{j=1}^{\infty}\frac{1}{s_j^2+\varepsilon(\mu-i\omega)}=$$
$$=\sum_{j=1}^{\infty}\frac{1}{[a_j^2+\varepsilon(\mu-i\omega)]}\left[1-\frac{2a_j\alpha_j+\alpha_j^2}{a_j^2+\varepsilon(\mu-i\omega)}\right]^{-1}=$$
$$=\sum_{j=1}^{\infty}\frac{1}{a_j^2+\varepsilon(\mu-i\omega)}+\sum_{j=1}^{\infty}\left\{\frac{2a_j\alpha_j+\alpha_j^2}{[a_j^2+\varepsilon(\mu-i\omega)]^2}+\right.$$
$$\left.+\frac{(2a_j\alpha_j+\alpha_j^2)^2}{[a_j^2+\varepsilon(\mu-i\omega)]^3}+\frac{(2a_j\alpha_j+\alpha_j^2)^2}{[a_j^2+\varepsilon(\mu-i\omega)]^4}+\cdots\right\}. \quad (5.6)$$

Next, the leading term in the first sum on the right can be isolated as follows:

$$\sum_{j=1}^{\infty}\frac{1}{a_j^2+\varepsilon(\mu-i\omega)}=$$
$$=\frac{1}{\pi}\sum_{j=1}^{\infty}\frac{\Delta a_j}{a_j^2+\varepsilon(\mu-i\omega)}\approx\frac{1}{\pi}\int_0^{\infty}\frac{da}{a^2+\varepsilon(\mu-i\omega)}=$$
$$=\frac{1}{2\sqrt{\varepsilon(\mu-i\omega)}} \qquad (\Delta a_j=a_j-a_{j-1}=\pi). \quad (5.7)$$

The leading term in the second sum on the right can be isolated in the same way. Using Eq. (5.5) we have

$$\sum_{j=1}^{\infty}\frac{2a_j\alpha_j}{[a_j^2+\varepsilon(\mu-i\omega)]}=2\sum_{j=1}^{\infty}\frac{1+{}^2\!/_3\,a_j^{-2}+{}^{13}\!/_{15}\,a_j^{-4}+\ldots}{[a_j^2+\varepsilon(\mu-i\omega)]^2}\approx$$
$$\approx\frac{2}{\pi}\sum_{j=1}^{\infty}\frac{\Delta a_j}{[a_j^2+\varepsilon(\mu-i\omega)]^2}\approx$$
$$\approx\frac{2}{\pi}\int_0^{\infty}\frac{da}{[a^2+\varepsilon(\mu-i\omega)]^2}=\frac{1}{2[\varepsilon(\mu-i\omega)]^{3/2}}$$

If

$$\int_0^{\infty}\frac{da}{a^2+\varepsilon(\mu-i\omega)}$$

is regarded as the area under the curve

$$y=\frac{1}{a^2+\varepsilon(\mu-i\omega)}, \qquad \text{and} \quad \sum_{j=1}^{\infty}\frac{\Delta a_j}{a_j^2+\varepsilon(\mu-i\omega)}$$

is taken to be the total area of strips of base length Δa_j and height

$$y_i=\frac{1}{a_j^2+\varepsilon(\mu-i\omega)}$$

then the leading term of the difference between them can be regarded as the sum of the series consisting of the areas of the right-angled triangles with sides Δa_j and $y_{j-1}-y_j$. We then have

$$\int_0^{\infty}\frac{da}{a^2+\varepsilon(\mu-i\omega)}-$$
$$-\sum_{j=1}^{\infty}\frac{\Delta a_j}{a_j^2+\varepsilon(\mu-i\omega)}\approx\frac{1}{2}\sum_{j=1}^{\infty}\Delta a_j\left(\frac{1}{a_{j-1}^2+\varepsilon(\mu-i\omega)}-\right.$$
$$\left.-\frac{1}{a_j^2+\varepsilon(\mu-i\omega)}\right)\approx\frac{\pi}{2}\int_0^{\infty}\frac{2a\,da}{a^2+\varepsilon(\mu-i\omega)}=$$
$$=\frac{\pi}{2\varepsilon(\mu-i\omega)}. \quad (5.8)$$

From Eqs. (5.7) and (5.8) we have, to within terms of the order of $\varepsilon^{-3/2}$,

$$\sum_{j=1}^{\infty}\frac{1}{s_j^2+\varepsilon(\mu-i\omega)}\approx\frac{1}{2}\left[\frac{1}{\sqrt{\varepsilon(\mu-i\omega)}}-\frac{1}{\varepsilon(\mu-i\omega)}\right] \quad (5.9)$$

Substituting Eq. (5.9) into (5.1), and returning to the variables λ and Ω_z, we obtain the approximate equation

$$(A - I)\lambda^2 + \frac{5I\lambda^2}{\sqrt{R^2/\nu(\lambda - i\Omega_z)}} - \frac{5I\lambda^2}{R^2/\nu(\lambda - i\Omega_z)} + Qa = 0. \quad (5.10)$$

The approximate equations for the frequencies corresponding to Eq. (4.3) can be obtained in the same way.

Equation (4.3) can be transformed so that it reads

$$\mu^2 - i\frac{C}{A}\omega\mu + 1 =$$
$$= 10\frac{I}{A}(\mu - i\omega)[\varepsilon\mu^2 S_4 - \varepsilon^2\mu^3 S_6 - \varepsilon^3\mu^4 S_8 - \ldots]. \quad (5.11)$$

For small ε the quantity μ can be sought in the form of the series (5.4). Substituting this series into Eq. (5.11), we can readily obtain the relation for μ_k.

For small ε^{-1} we have the following approximate equation for the frequencies:

$$(A - J)\lambda^2 + 5\frac{\sqrt{\nu}}{R}J\lambda^{3/2} - \left[i\Omega_z(C - J) + \frac{5\nu}{R^2}J\right]\lambda -$$
$$- i\frac{(C - I)\sqrt{\nu}}{R}\Omega_z\lambda^{1/2} + Qa + i\frac{5\nu}{R^2}\Omega_z J = 0.$$

A number of papers have appeared in recent years in which the motion of a body filled with a viscous fluid was investigated by asymptotic methods for large and small Reynolds numbers. We note that Eqs. (5.10) for $\Omega_z = 0$ transforms to the more accurate equation obtained by Krasnoshchekov [5] which contains the first three expansion terms.

The case of small Reynolds numbers was investigated by Chernous'ko [6]. If we use his method in the case of a spherical cavity, then the angular velocity of proper motion is described by an equation which is the same as Eq. (3.4) to within terms of the order of ν^{-2}.

For the case of slow rotation, frequency equation (3.7) will also be identical with the equation obtained by the method described in [6] to within terms of the order of Ω_z^2 and ν^{-2}. The somewhat unexpected result is that the equations are also the same in the case of rapid rotation which was considered here for the special solution and in [6] in the general case.

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